

ON THE MAXIMAL VOLUME OF THREE-DIMENSIONAL HYPERBOLIC COMPLETE ORTHOSCHEMES

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ABSTRACT. A three-dimensional orthoscheme is defined as a tetrahedron whose base is a right-angled triangle and an edge joining the apex and a non-right-angled vertex is perpendicular to the base. A generalization, called complete orthoschemes, of orthoschemes is known in hyperbolic geometry. Roughly speaking, complete orthoschemes consist of three kinds of polyhedra; either compact, ideal or truncated. We consider a particular family of hyperbolic complete orthoschemes, which share the same base. They are parametrized by the “height”, which represents how far the apex is from the base. We prove that the volume attains maximal when the apex is ultraideal in the sense of hyperbolic geometry, and that such a complete orthoscheme is unique in the family.

1. INTRODUCTION

In [Ke], Kellerhals wrote “the most basic objects in polyhedral geometry are orthoschemes”, and she gave a formula to calculate the volumes of complete orthoschemes in the three-dimensional hyperbolic space. What we discuss here is the existence and the uniqueness of the maximal volume of a family of complete orthoschemes parametrized by the “height”.

Consider a family of pyramids in Euclidean space with a fixed base polygon and the locus of apexes perpendicular to the base polygon. The volumes of pyramids strictly increases when the height increases, because pyramids strictly increases as a set. By the same reason, this phenomenon holds true for such a family of pyramids in hyperbolic space. In contrast to the Euclidean case, the volume approaches to a finite value. Furthermore, in hyperbolic space the apex can “run out” the space. Then we can still obtain finite volume hyperbolic polyhedron by *truncation* with respect to the apex. The volume converges to zero as the vertex goes away from the space. So it is an interesting question when the volume becomes maximum.

As is mentioned above, one of the most fundamental one among all such pyramids is the orthoscheme. An orthoscheme is a kind of simplex which has particular orthogonality among its faces. Let P_0, P_1, P_2 and P_3 be the vertices of a simplex R in the three-dimensional hyperbolic space. We denote by P_iP_j the edge spanned by P_i and P_j , and by $P_iP_jP_k$ the face spanned by P_i, P_j and P_k . Such a simplex R is called an *orthoscheme* (in the ordinary sense) if the edge P_0P_1 is perpendicular to the face $P_1P_2P_3$ and the face $P_0P_1P_2$ is orthogonal to P_2P_3 . In other words, an orthoscheme is a tetrahedron with a right-angled triangle $P_0P_1P_2$ as its base and

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an edge joining the apex and a non-right-angled vertex, say P_2 , is perpendicular to the base. Vertices P_0 and P_3 are called the *principal vertices* of R . Its precise definition will be given in Section 3.

Though orthoschemes are also considered in Euclidean or spherical spaces, in hyperbolic space the ordinary orthoschemes are extended to the so-called *complete orthoschemes*. Let B^3 be the open unit ball in the three-dimensional Euclidean space \mathbb{R}^3 centered at the origin. The set B^3 can be regarded as the so-called *projective ball model* of the three-dimensional hyperbolic space. Any tetrahedron in hyperbolic space appears as a Euclidean tetrahedron in B^3 . If one or both principal vertices of an orthoscheme R lie in the boundary of B^3 , the set $R \cap B^3$ is called an *ideal polyhedron*, which is not bounded in hyperbolic space, while its volume is finite. Take one step further and we allow principal vertices to be in the exterior of B^3 . The volume of $R \cap B^3$ is no longer finite, but there is a canonical way to delete ends of $R \cap B^3$ with infinite volume so that we obtain a polyhedron of finite volume, called a *truncated polyhedron*. Complete orthoschemes are, roughly speaking, either compact, ideal or truncated orthoschemes. The precise definitions of complete orthoschemes and truncation will also be given in Section 3.

What we study in this paper is the maximal volume of a family of complete orthoschemes with one parameter. Consider a family of complete orthoschemes that share the same base $P_0P_1P_2$. We allow the vertex P_0 to be in the exterior of B^3 . In this case the base $P_0P_1P_2$ means the truncated polygon obtained from the triangle with vertices P_0 , P_1 and P_2 . Such a family of complete orthoschemes is parametrized by the hyperbolic length of the edge P_2P_3 when P_3 is in B^3 . When the hyperbolic length increases, the orthoscheme strictly increases as a set, which means the volume also increases with respect to the function of the hyperbolic length. This phenomenon holds until the vertex P_3 lies in the boundary ∂B^3 of B^3 . The hyperbolic length of P_2P_3 is “beyond” the infinity when P_3 is in the exterior of B^3 , but we have a complete orthoscheme with finite volume by truncation. Instead of the hyperbolic length, using the Euclidean length of P_2P_3 , which we mentioned as “height” in the first paragraph, we can parametrize the family even if P_3 is in the complement of B^3 . The complete orthoscheme approaches the empty set when P_3 goes far away from B^3 . The family thus has maximal volume complete orthoschemes, which arise when P_3 lies in the complement of B^3 .

As a toy model, let us consider the same phenomenon for the two-dimensional orthoschemes, namely hyperbolic triangle $P_0P_1P_2$ with right angle at P_1 . Take a family of complete orthoschemes parametrized by the “height” of P_1P_2 . The area strictly increases when P_2 approaches to the boundary of B^2 , the projective disc model of the two-dimensional hyperbolic space. The area attains maximal when P_2 lies in ∂B^2 . When P_2 is in the exterior of B^2 , the area decreases, but not necessarily monotonically. These facts are summarized as Theorem 2 in the appendix.

One may expect that the same phenomenon happens for three-dimensional complete orthoschemes. Is the volume attains maximal at least when P_3 is in ∂B^3 ? Does the volume decrease when P_3 goes far away from B^3 ? Our main result, which is Theorem 1 in Section 5, answers both of the questions negatively.

2. PRELIMINARIES OF HYPERBOLIC GEOMETRY

There are several models to introduce hyperbolic geometry. Among them we use the *hyperboloid model* to calculate lengths and angles with respect to the hyperbolic

metric, and use the *projective ball model* to define complete orthoschemes. Definitions of these two models, together with formulae to calculate hyperbolic lengths and hyperbolic angles, are explained in this section. See [Ra] for basic references on hyperbolic geometry.

As a set, the hyperboloid model H_T^+ of the three-dimensional hyperbolic space is defined as a subset of the four-dimensional Euclidean space \mathbb{R}^4 by

$$H_T^+ := \{ \mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1 \text{ and } x_0 > 0 \},$$

where $\langle \cdot, \cdot \rangle$, called the *Lorentzian inner product*, is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$

for any $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$ in \mathbb{R}^4 . The restriction of the quadratic form induced from the Lorentzian inner product to the tangent spaces of H_T^+ is positive definite and gives a Riemannian metric on H_T^+ , which is constant curvature of -1 . The set H_T^+ together with this metric gives the *hyperboloid model* of the three-dimensional hyperbolic space.

Associated with H_T^+ , there are two important subsets of \mathbb{R}^4 :

$$H_S := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1 \}, \quad L^+ := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0 \text{ and } x_0 > 0 \}.$$

Every point \mathbf{u} in H_S corresponds to a half-space

$$R_{\mathbf{u}} := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq 0 \},$$

bounded by a plane

$$P_{\mathbf{u}} := \{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle = 0 \}.$$

The intersection $P_{\mathbf{u}} \cap H_T^+$ is a geodesic plane with respect to the hyperbolic metric. If \mathbf{u} is taken from L^+ , the set $R_{\mathbf{u}}$ is defined as

$$R_{\mathbf{u}} := \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle \leq -\frac{1}{2} \right\}.$$

The intersection $R_{\mathbf{u}} \cap H_T^+$ is called a *horoball*. The intersection of the boundary

$$P_{\mathbf{u}} := \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{u} \rangle = -\frac{1}{2} \right\}$$

of $R_{\mathbf{u}}$ and H_T^+ is called a *horosphere*.

The Lorentzian inner product is also used to calculate distances and angles with respect to the hyperbolic metric. The details of the following results are explained in §3.2 of [Ra]. Let \mathbf{u} be a point in H_T^+ and let \mathbf{v} be taken from H_S with $\mathbf{u} \in R_{\mathbf{v}}$, then the hyperbolic distance ℓ between \mathbf{u} and the geodesic plane $P_{\mathbf{v}}$ is calculated by

$$(2.1) \quad \sinh \ell = -\langle \mathbf{u}, \mathbf{v} \rangle.$$

Suppose that \mathbf{u} is in L^+ and \mathbf{v} is in H_S with $\mathbf{u} \in R_{\mathbf{v}}$. Let ℓ be the signed hyperbolic distance between the horosphere $P_{\mathbf{u}} \cap H_T^+$ and the geodesic plane $P_{\mathbf{v}} \cap H_T^+$. The sign is defined to be positive if the horosphere and the geodesic plane do not intersect, otherwise negative. Then the signed distance ℓ is calculated by

$$(2.2) \quad \frac{e^{\ell}}{2} = -\langle \mathbf{u}, \mathbf{v} \rangle.$$

If both \mathbf{u} and \mathbf{v} are taken from H_S with $\mathbf{u} \in R_{\mathbf{v}}$ and $\mathbf{v} \in R_{\mathbf{u}}$, then there are three possibilities: $R_{\mathbf{u}} \cap R_{\mathbf{v}}$ intersects H_T^+ , intersects L^+ or does not intersect both

H_T^+ and L^+ . The first case means that the geodesic planes $P_u \cap H_T^+$ and $P_v \cap H_T^+$ intersect and form a corner $R_u \cap R_v \cap H_T^+$. The hyperbolic dihedral angle θ between these geodesic planes measured in this corner is calculated by

$$(2.3) \quad \cos \theta = -\langle \mathbf{u}, \mathbf{v} \rangle.$$

The third case means that the geodesic planes $P_u \cap H_T^+$ and $P_v \cap H_T^+$ are *ultraparallel*, meaning that they do not intersect in H_T^+ and there is a unique geodesic line in H_T^+ which is perpendicular to these geodesic planes. The hyperbolic length ℓ of the segment between these planes is calculated by

$$(2.4) \quad \cosh \ell = -\langle \mathbf{u}, \mathbf{v} \rangle.$$

The second case is regarded as the first case with hyperbolic dihedral angle 0 or the third case with the hyperbolic distance 0. Geodesic planes in this case are called *parallel* in hyperbolic space.

The *projective ball model* B^3 is another model of the three-dimensional hyperbolic space, which is induced from H_T^+ . Let \mathcal{P} be the radial projection from $\mathbb{R}^4 - \{ \mathbf{x} \in \mathbb{R}^4 \mid x_0 = 0 \}$ to the affine hyperplane $\mathbf{P}_1 := \{ \mathbf{x} \in \mathbb{R}^4 \mid x_0 = 1 \}$ along the ray from the origin \mathbf{o} of \mathbb{R}^4 . The projection \mathcal{P} is a homeomorphism on H_T^+ to the three-dimensional open unit ball B^3 in \mathbf{P}_1 centered at $(1, 0, 0, 0)$. A metric is induced on B^3 from H_T^+ by the projection. With this metric B^3 is called the *projective ball model* of the three-dimensional hyperbolic space. The projection \mathcal{P} also induces the mapping from $\mathbb{R}^4 - \{ \mathbf{o} \}$ to the three-dimensional real projective space $\mathbb{R}P^3$, which is defined to be the union of \mathbf{P}_1 and the set of lines in the affine hyperplane $\{ \mathbf{x} \in \mathbb{R}^4 \mid x_0 = 0 \}$ through \mathbf{o} . In contrast to ordinary points in B^3 , points in the set ∂B^3 of the boundary of B^3 are called *ideal*, and points in the exterior of B^3 are called *ultraideal*. We often regard B^3 as the unit open ball centered at the origin in \mathbb{R}^3 .

We mention important properties of B^3 to be used in the definition of complete orthoschemes in the next section. First, every geodesic plane in B^3 is given as the intersection of a Euclidean plane and B^3 . This is because every geodesic plane in H_T^+ is defined as the intersection of H_T^+ and a linear subspace of \mathbb{R}^4 of dimension three, and a geodesic plane in B^3 is the image of that in H_T^+ by the radial projection. The projection \mathcal{P} thus gives a correspondence between points in the exterior of B^3 in $\mathbb{R}P^3$ and the geodesic planes of B^3 . We call $\mathcal{P}(\mathbf{u})$ for $\mathbf{u} \in H_S$ the *pole* of the plane $\mathcal{P}(P_u)$ or the geodesic plane $\mathcal{P}(P_u \cap H_T^+)$. Conversely, we call $\mathcal{P}(P_u)$ the *polar plane* of $\mathcal{P}(\mathbf{u})$, and we call $\mathcal{P}(P_u \cap H_T^+)$ the *polar geodesic planes* of $\mathcal{P}(\mathbf{u})$. If $\mathcal{P}(P_u \cap H_T^+)$ does not pass through the origin of B^3 , then its pole is given as the apex of a circular cone which is tangent to ∂B^3 and has the base circle $P_u \cap \partial B^3$. The second important property is that, for a given geodesic plane, say P , in B^3 , every plane or line which passes through the pole of P is orthogonal to P in B^3 . This is proved by using Equation (2.3).

3. COMPLETE ORTHOSCHEMES

Following [Ke] we introduce complete orthoschemes. As is mentioned in the introduction, an (ordinary) *orthoscheme* in the three-dimensional hyperbolic space is a tetrahedron with vertices P_0, P_1, P_2 and P_3 which satisfies that P_0P_1 is perpendicular to $P_1P_2P_3$ and that $P_0P_1P_2$ is orthogonal to P_2P_3 . The vertices P_0 and P_3 are called *principal vertices*.

Complete orthoschemes are a generalization of ordinary orthoschemes by allowing one or both principal vertices to be ideal or ultraideal. Take B^3 as our favorite model of the hyperbolic space in what follows. As a set, any orthoscheme in the ordinary sense are given as a Euclidean tetrahedron in B^3 . When one or both principal vertices are ideal, the tetrahedron as a set in the hyperbolic space is no more bounded, but still has finite volume. We allow to call such tetrahedra ordinary orthoschemes.

Further generalization of orthoschemes is explained via *truncation* of ultraideal vertices. Suppose a vertex v of a tetrahedron R is ultraideal. Let T be the half-space bounded by the polar plane of v with $v \notin T$. *Truncation* of R with respect to v is defined as an operation to obtain a polyhedron $R \cap T$. If v is close enough to ∂B^3 , then $R \cap T$ is non-empty.

Truncation is also explained by using the hyperboloid model. The inverse image of v for \mathcal{P} on H_S consists of two points. Each of them gives a half-space in \mathbb{R}^4 , and one of them corresponds to the inverse image of T . In this sense there is a one-to-one correspondence between half-spaces in B^3 and points in H_S . The point in H_S corresponding to v with respect to T in the sense above is called the *proper* inverse image of v for truncation of R . This correspondence will be used to calculate hyperbolic lengths of edges and hyperbolic dihedral angles between faces of complete orthoschemes.

When one of the principal vertices, say P_3 , is ultraideal and P_0 is not (i.e, ordinal or ideal), we have a polyhedron with finite volume by truncation with respect to P_3 . Such a polyhedron is called a *simple frustum* with ultraideal vertex P_3 . We remark that the vertices P_0 , P_1 and P_2 are simultaneously deleted by truncation when P_3 is far away from B^3 , since both the polar geodesic plane of P_3 and the triangle $P_0P_1P_2$ are orthogonal to P_2P_3 in B^3 .

Suppose both P_0 and P_3 are ultraideal. There are three possibilities: the polar planes of P_0 and P_3 intersect in B^3 , they are parallel, or they are ultraparallel. In the first case, the polyhedron we obtain by truncation is well known as a *Lambert cube*. See [Ke, Figure 2] for example. The edge P_0P_3 is deleted by truncation. In the third case, on the other hand, the polyhedron obtained by truncation still has the edge induced from P_0P_3 . We call this polyhedron a *double frustum*. The second case is the limiting situation of both the first and the third cases. We call polyhedra obtained in the second case *double frustum with an ideal vertex*.

As a summary, combinatorial types of complete orthoschemes are either

- ordinary orthoschemes, whose principal vertices are either ordinarily points or ideal points,
- simple frustums,
- double frustums possibly with an ideal vertex, or
- Lambert cubes.

4. THE SCHLÄFLI DIFFERENTIAL FORMULA

Kellerhals obtained formulae to calculate volumes of complete orthoschemes in [Ke]; the formula for Lambert cubes is given in Theorem III, and the formula for other kinds of complete orthoschemes is given in Theorem II. In both formulae, they are parametrized by the three non-right hyperbolic dihedral angles. Under the same setting used in Section 3, we denote by $\theta_{i,j}$ the hyperbolic dihedral angle between faces opposite to P_i and P_j . When a complete orthoscheme is a Lambert

cube, the geodesic planes containing faces opposite to P_1 and P_2 are ultraparallel. In this case $\theta_{1,2}$ is defined to be the hyperbolic dihedral angle between the polar geodesic planes of the vertex P_0 and P_3 . In this sense the formulae are parametrized by $\theta_{0,1}$, $\theta_{1,2}$ and $\theta_{2,3}$.

Kellerhals used the *Schläfli differential formula* to obtain these volume formulae. The volume formulae are not used directly in our arguments; what we will use is the fact that the formulae are parametrized by the three non-right hyperbolic dihedral angles. On the other hand, the Schläfli differential formula itself plays an important role in our arguments.

The Schläfli differential formula gives an expression of the differential form of the volume function with respect to the hyperbolic lengths of edges and hyperbolic dihedral angles between faces. As is given in Theorem I in [Ke], the differential form dV of the volume function V of any complete orthoschemes is expressed as

$$dV = -\frac{1}{2}(\ell_{0,1} d\theta_{0,1} + \ell_{1,2} d\theta_{1,2} + \ell_{2,3} d\theta_{2,3}),$$

where $\ell_{i,j}$ is the hyperbolic length of the edge $P_i P_j$ if both P_i and P_j are points in B^3 , $\ell_{i,j}$ is the hyperbolic distance between P_i and the polar geodesic plane of P_j if P_i is a point in B^3 and P_j lies in the exterior of B^3 , and $\ell_{i,j}$ is the hyperbolic distance between the polar geodesic planes of P_i and P_j if both P_i and P_j lie in the exterior of B^3 . If a complete orthoscheme is a Lambert cube, then $\ell_{0,3}$ is taken as the hyperbolic length of the edge obtained as the intersection of the polar geodesic planes of P_0 and P_3 . If one of P_0 and P_3 is ideal, then the edges with the ideal vertex as an endpoint have infinite hyperbolic lengths. In this case we take any horosphere centered at the ideal vertex, and each infinite length is replaced by the signed hyperbolic distance between the other endpoint and the horosphere. As is mentioned in the concluding remarks in [Mi], the Schläfli differential formula is still valid by this treatment.

As a result, the Schläfli differential formula is applicable to any kind of complete orthoschemes. We use the formula as the equation

$$(4.1) \quad \frac{\partial V}{\partial \theta_{i,j}} = -\frac{1}{2} \ell_{i,j}$$

for $(i,j) = (0,1), (1,2), (2,3)$. This equation plays a key role in the proof of Theorem 1.

5. MAIN RESULT

Suppose B^3 lies in the xyz -coordinate space of \mathbb{R}^3 . By the action of an isometry, any ordinary orthoscheme can be put as the vertex P_0 is in the positive quadrant of the xy -plane, the vertex P_1 is on the positive part of the y -axis, the vertex P_2 is the origin, and the vertex P_3 is on the positive part of the z -axis. Such orthoschemes are parametrized by (h, r, θ) , where h is the z -coordinate of P_3 , i.e., the Euclidean distance between P_2 and P_3 , r is the Euclidean distance between P_0 and P_2 , and θ is the Euclidean angle between edges $P_0 P_2$ and $P_1 P_2$.

When we regard such an orthoscheme as a tetrahedron with base $P_0 P_1 P_2$, the z -coordinate h of P_3 is the “height” of the tetrahedron. What we study in this paper is a family of complete orthoschemes parametrized by the “height”. For fixed r and θ , we have a one-parameter family $\{R_{r,\theta}(h)\}_{0 < h \leq 1}$ of ordinary orthoschemes

parametrized by h . This family is extended even when $h \geq 1$ and/or $r \geq 1$ with $r \cos \theta < 1$, if we mean $R_{r,\theta}(h)$ a complete orthoscheme.

Let $V_{r,\theta}(h)$ be the hyperbolic volume of $R_{r,\theta}(h)$. By the volume formulae, the function $V_{r,\theta}$ is continuous on $[0, +\infty)$ and piecewise differentiable on the intervals each of which corresponds to a combinatorial type of complete orthoschemes given at the end of Section 3. When h increases in value approaching 1, the orthoscheme also increases as a set. So $V_{r,\theta}(h)$ strictly increases in value approaching $V_{r,\theta}(1)$ as h approaches 1 from below. When h approaches positive infinity $+\infty$, the sequence $R_{r,\theta}(h)$ of complete orthoschemes converges to the base $P_0P_1P_2$; the complete orthoschemes are always ordinary ones when $0 < r \leq 1$, and the complete orthoschemes changes into Lambert cubes from double frustums when $r > 1$. In any case $V_{r,\theta}(h)$ converges to 0 as h approaches $+\infty$.

Based on these observations, we have set the following questions. For a given one-parameter family $\{R_{r,\theta}(h)\}_{h>0}$ of complete orthoschemes, does the function $V_{r,\theta}$ attain maximal when P_3 is in ∂B^3 ? Is $V_{r,\theta}$ strictly decreasing on $(1, +\infty)$? The next theorem, which is the main result of this paper, answers both of the questions negatively.

Theorem 1. *For any $r > 0$ and $0 < \theta < \pi/2$ with $r \cos \theta < 1$, the volume $V_{r,\theta}(h)$ of $R_{r,\theta}(h)$ attains maximal for some $h \in (1, +\infty)$. Furthermore, the maximal volume is unique for any r and θ , and it is given before $R_{r,\theta}(h)$ becomes a Lambert cube.*

The outline of the proof is as follows. Using the Schläfli differential formula, we can calculate $dV_{r,\theta}(h)/dh$ for each combinatorial types of $R_{r,\theta}(h)$. Since $V_{r,\theta}$ is a strictly increasing function on $[0, 1]$, proving $\lim_{h \downarrow 1} dV_{r,\theta}(h)/dh > 0$ tells us that the function $V_{r,\theta}$ attains maximal for some $h \in (1, +\infty)$. The uniqueness of such h is induced from the uniqueness of the solution of the equation $dV_{r,\theta}(h)/dh = 0$ on $(1, +\infty)$.

6. PROOF OF THE MAIN RESULT

Our proof of Theorem 1 is organized as follows. After confirming the correspondence between combinatorial types of complete orthoschemes and conditions of parameters h , r and θ , we first obtain suitable inverse images of vertices of $R_{r,\theta}(h)$ for \mathcal{P} . These are used to calculate hyperbolic lengths and hyperbolic dihedral angles appearing in the Schläfli differential formula. Under each of conditions of parameters, we prove that the volume function $V_{r,\theta}$ with respect to h attains maximal on $(1, +\infty)$, and that such h is unique. For $r > 1$, we also prove that $V_{r,\theta}$ does not attain maximal if $R_{r,\theta}(h)$ is a Lambert cube.

6.1. Proper inverse images of the vertices. By the definition of $R_{r,\theta}(h)$, the coordinates of the vertices are

$$\begin{aligned} P_0 &= (r \sin \theta, r \cos \theta, 0), & P_1 &= (0, r \cos \theta, 0), \\ P_2 &= (0, 0, 0), & P_3 &= (0, 0, h), \end{aligned}$$

where $0 < \theta < \pi/2$. As is mentioned after Theorem 1, it is enough to assume that $h > 1$ in what follows. A complete orthoscheme $R_{r,\theta}(h)$ is a simple frustum if $0 < r < 1$, and a simple frustum with ideal vertex P_0 if $r = 1$. When $r > 1$, we always assume $r \cos \theta < 1$ so that P_1 is in B^3 . Under these assumptions, a complete orthoscheme $R_{r,\theta}(h)$ with $r > 1$ is either a double frustum, a double frustum with an ideal vertex, or a Lambert cube. These are distinguished via the Euclidean

distance between the origin of \mathbb{R}^3 and the edge P_0P_3 ; $R_{r,\theta}(h)$ is a double frustum, a double frustum with an ideal vertex, or a Lambert cube if and only if the Euclidean distance is less than, equal to, or greater than 1 respectively. Since the Euclidean distance is $hr/\sqrt{r^2+h^2}$, we have that these are equivalent to $h < r/\sqrt{r^2-1}$, $h = r/\sqrt{r^2-1}$, or $h > r/\sqrt{r^2-1}$ respectively. The inequality $hr/\sqrt{r^2+h^2} < 1$ is also equivalent to $(1-r^2)h^2 + r^2 > 0$ without the assumption that $r > 1$. We note that this inequality always holds for any $h > 0$ and $0 < r \leq 1$.

As a summary, complete orthoschemes $R_{r,\theta}(h)$ are parametrized by (h, r, θ) , and with $h > 1$ and $0 < \theta < \pi/2$, and

- when $0 < r \leq 1$, complete orthoschemes $R_{r,\theta}(h)$ are simple frustums with $(1-r^2)h^2 + r^2 > 0$,
- when $r > 1$ with $r \cos \theta < 1$ and $h \leq r/\sqrt{r^2-1}$, complete orthoschemes $R_{r,\theta}(h)$ are double frustums (possibly with an ideal vertex), and
- when $r > 1$ with $r \cos \theta < 1$ and $h > r/\sqrt{r^2-1}$, complete orthoschemes $R_{r,\theta}(h)$ are Lambert cubes.

We next give the proper inverse images of these vertices for \mathcal{P} . When a vertex is in B^3 , its inverse image for \mathcal{P} must be chosen in H_T^+ , which is uniquely determined. When a vertex is in the exterior of B^3 , its inverse image is chosen to be proper inverse image in the sense of truncation. Finally, when a vertex is in ∂B^3 , we choose its proper inverse image as any element in the inverse for \mathcal{P} , which is a subset in L^+ . Let \mathbf{p}_i be the proper inverse image of P_i in this sense. The coordinates of \mathbf{p}_i are then as follows:

- (1) When $0 < r < 1$, we have

$$\begin{aligned}\mathbf{p}_0 &= \frac{1}{\sqrt{1-r^2}}(1, r \sin \theta, r \cos \theta, 0), \\ \mathbf{p}_1 &= \frac{1}{\sqrt{1-r^2 \cos^2 \theta}}(1, 0, r \cos \theta, 0), \\ \mathbf{p}_2 &= (1, 0, 0, 0), \\ \mathbf{p}_3 &= \frac{1}{\sqrt{h^2-1}}(1, 0, 0, h).\end{aligned}$$

- (2) When $r = 1$, the coordinates of \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 are the same as in the first case and

$$\mathbf{p}_0 = (1, \sin \theta, \cos \theta, 0).$$

- (3) When $r > 1$, the coordinates of \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 are the same as in the first case, and

$$\mathbf{p}_0 = \frac{1}{\sqrt{r^2-1}}(1, r \sin \theta, r \cos \theta, 0).$$

The inverse image of the pole of a geodesic plane in B^3 consists of two points in H_S . For each (ordinary) face of an orthoscheme $R_{r,\theta}(h)$, we choose the inverse image of the pole in H_S so that the half-space defined by this inverse image contains $R_{r,\theta}(h)$. Let \mathbf{u}_i be the inverse image of the pole of the face $P_jP_kP_l$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$ in this sense. In other words, \mathbf{u}_i is a point in H_S where $R_{\mathbf{u}_i}$ contains $R_{r,\theta}(h)$ and

P_{u_i} contains $P_j P_k P_l$. For any r , the coordinates of u_i are as follows:

$$\begin{aligned} u_0 &= (0, -1, 0, 0), \\ u_1 &= (0, \cos \theta, -\sin \theta, 0), \\ u_2 &= \frac{1}{\sqrt{(1-r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} (h r \cos \theta, 0, h, r \cos \theta), \\ u_3 &= (0, 0, 0, -1). \end{aligned}$$

6.2. The maximal value of $V_{r,\theta}$ and its uniqueness with respect to h . We focus on the derivative $dV_{r,\theta}(h)/dh$ to prove that $V_{r,\theta}$ attains maximal on $(1, +\infty)$, as well as its uniqueness.

We first confirm that the function $V_{r,\theta}$ is piecewise differentiable with respect to h in general.

We first suppose that $r \leq 1$. By the Schläfli differential formula, the function $V_{r,\theta}$ is differentiable with respect to the hyperbolic dihedral angles $\theta_{0,1}$, $\theta_{1,2}$ and $\theta_{2,3}$. By the expression of the coordinates of u_i and p_i for $i = 0, 1, 2, 3$ together with Equation (2.3), these angles are given as smooth functions with respect to h . By the chain rule, $V_{r,\theta}$ is thus differentiable with respect to h . In particular $V_{r,\theta}$ is continuous on $[0, +\infty)$.

If $r > 1$, then there are two combinatorial types of $R_{r,\theta}(h)$; a double frustum or a Lambert cube. The function $V_{r,\theta}$ is not only continuous but also piecewise differentiable on $[0, +\infty)$, for $V_{r,\theta}$ is differentiable on the intervals corresponding to each combinatorial types of $R_{r,\theta}(h)$ by the same argument used for $r \leq 1$.

Recall that the function $V_{r,\theta}$ is continuous on $[0, +\infty)$, strictly increasing on $[0, 1]$ and has its limit 0 as h approaches $+\infty$. So, to prove that $V_{r,\theta}$ attains maximal on $(1, +\infty)$, it is enough to prove that the limit of $dV_{r,\theta}(h)/dh$ is positive as h approaches to 1 from above. The uniqueness of the maximal value of $V_{r,\theta}$ is induced from the fact that the solution of $dV_{r,\theta}(h)/dh = 0$ is at most one on $(1, +\infty)$.

Applying the chain rule and we have

$$\frac{dV_{r,\theta}(h)}{dh} = \frac{\partial V_{r,\theta}(h)}{\partial \theta_{0,1}} \frac{d\theta_{0,1}}{dh} + \frac{\partial V_{r,\theta}(h)}{\partial \theta_{1,2}} \frac{d\theta_{1,2}}{dh} + \frac{\partial V_{r,\theta}(h)}{\partial \theta_{2,3}} \frac{d\theta_{2,3}}{dh}.$$

The parameter $\theta_{i,j}$ defined in Section 4 is the hyperbolic dihedral angle between the polar geodesic planes of $\mathcal{P}(u_i)$ and $\mathcal{P}(u_j)$. In other words, $\theta_{i,j}$ is the hyperbolic dihedral angle along the edge $P_k P_l$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$. As is mentioned in the first paragraph of Section 4, if $R_{r,\theta}(h)$ is a Lambert cube, then $\theta_{1,2}$ is taken as the hyperbolic dihedral angle between the polar geodesic planes of P_0 and P_3 .

The Schläfli differential formula are used to calculate partial derivatives appeared in the equation above. By Equation (4.1) we have

$$\frac{\partial V_{r,\theta}(h)}{\partial \theta_{1,2}} = -\frac{1}{2} \ell_{0,3}, \quad \frac{\partial V_{r,\theta}(h)}{\partial \theta_{2,3}} = -\frac{1}{2} \ell_{0,1},$$

where $\ell_{i,j}$ is the hyperbolic length with respect to the edge $P_i P_j$ defined in Section 4. Furthermore, the hyperbolic dihedral angle $\theta_{0,1}$, which coincides with the Euclidean angle θ by the definition of $R_{r,\theta}(h)$, is constant with respect to h , meaning that $d\theta_{0,1}/dh = 0$. We thus have

$$(6.1) \quad \frac{dV_{r,\theta}(h)}{dh} = -\frac{1}{2} \left(\ell_{0,3} \frac{d\theta_{1,2}}{dh} + \ell_{0,1} \frac{d\theta_{2,3}}{dh} \right).$$

We divide the remaining argument into three cases according to the value of r .

Case (1): *single frustums with ordinary vertex* P_0 , i.e., $0 < r < 1$. By Equations (2.1) we have

$$\begin{aligned}
 \ell_{0,3} &= \operatorname{arcsinh}(-\langle \mathbf{p}_0, \mathbf{p}_3 \rangle) \\
 &= \operatorname{arcsinh} \frac{1}{\sqrt{1-r^2} \sqrt{h^2-1}} \\
 &= \log \left(\frac{1}{\sqrt{1-r^2} \sqrt{h^2-1}} + \sqrt{\left(\frac{1}{\sqrt{1-r^2} \sqrt{h^2-1}} \right)^2 + 1} \right) \\
 (6.2) \quad &= \log \frac{\sqrt{(1-r^2)h^2+r^2}+1}{\sqrt{1-r^2} \sqrt{h^2-1}},
 \end{aligned}$$

and by Equation (2.3) we have

$$\begin{aligned}
 \theta_{1,2} &= \arccos(-\langle \mathbf{u}_1, \mathbf{u}_2 \rangle) \\
 &= \arccos \frac{h \sin \theta}{\sqrt{(1-r^2 \cos^2 \theta)h^2 + r^2 \cos^2 \theta}}, \\
 \theta_{2,3} &= \arccos(-\langle \mathbf{u}_2, \mathbf{u}_3 \rangle) \\
 &= \arccos \frac{r \cos \theta}{\sqrt{(1-r^2 \cos^2 \theta)h^2 + r^2 \cos^2 \theta}}.
 \end{aligned}$$

Derivatives of hyperbolic dihedral angles with respect to h are obtained as follows:

$$(6.3) \quad \frac{d\theta_{1,2}}{dh} = \frac{-r^2 \sin \theta \cos \theta}{\{(1-r^2 \cos^2 \theta)h^2 + r^2 \cos^2 \theta\} \sqrt{(1-r^2)h^2 + r^2}},$$

$$(6.4) \quad \frac{d\theta_{2,3}}{dh} = \frac{r \sqrt{1-r^2 \cos^2 \theta} \cos \theta}{(1-r^2 \cos^2 \theta)h^2 + r^2 \cos^2 \theta}.$$

Substitute Equations (6.2), (6.3) and (6.4) to Equation (6.1) and we have

$$\frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left(-\frac{d\theta_{1,2}}{dh} \right) \left(F(h) - \frac{1}{2} \log(1-r^2) \right),$$

where

$$(6.5) \quad F(h) := \log \frac{\sqrt{(1-r^2)h^2+r^2}+1}{\sqrt{h^2-1}} - C \sqrt{(1-r^2)h^2+r^2}$$

and $C := \ell_{0,1} \sqrt{1-r^2 \cos^2 \theta} / (r \sin \theta)$.

Since

$$\lim_{h \downarrow 1} \left(-\frac{d\theta_{1,2}}{dh} \right) = r^2 \sin \theta \cos \theta, \quad \lim_{h \downarrow 1} F(h) = +\infty,$$

we have

$$\begin{aligned}
 \lim_{h \downarrow 1} \frac{dV_{r,\theta}(h)}{dh} &= \frac{1}{2} (r^2 \sin \theta \cos \theta) \left(+\infty - \frac{1}{2} \log(1-r^2) \right) \\
 &= +\infty,
 \end{aligned}$$

which implies that $V_{r,\theta}$ attains maximal for some $h \in (1, +\infty)$.

This result together with $\lim_{h \uparrow +\infty} V_{r,\theta}(h) = 0$ implies that the uniqueness of the maximal value of the function $V_{r,\theta}$ with respect to h is proved by showing that the

equation $dV_{r,\theta}(h)/dh = 0$ has at most one solution on $(1, +\infty)$. Since $d\theta_{1,2}/dh \neq 0$ on $(1, +\infty)$ by Equation (6.3), we have

$$(6.6) \quad \left\{ h \in (1, +\infty) \mid \frac{dV_{r,\theta}(h)}{dh} = 0 \right\} \\ = \left\{ h \in (1, +\infty) \mid F(h) - \frac{1}{2} \log(1 - r^2) = 0 \right\}.$$

Since

$$(6.7) \quad \frac{d}{dh} \left(F(h) - \frac{1}{2} \log(1 - r^2) \right) = - \frac{h}{(h^2 - 1) \sqrt{(1 - r^2)h^2 + r^2}} G(h),$$

where $G(h) := C(1 - r^2)(h^2 - 1) + 1$, is negative on $(1, +\infty)$, the function $F(h) - (1/2)\log(1 - r^2)$ is strictly monotonic with respect to h . This implies that the number of elements in the right-hand side set of Equation (6.6) is at most one, so is the left-hand side.

Case (2): single frustums with ideal vertex P_0 , i.e., $r = 1$. Using Equation (2.2), we have

$$\ell_{0,3} = \log(-2 \langle \mathbf{p}_0, \mathbf{p}_3 \rangle) \\ = \log \frac{2}{\sqrt{h^2 - 1}}.$$

By Equations (6.3) and (6.4) with $r = 1$ and we have

$$-\frac{d\theta_{1,2}}{dh} = \frac{d\theta_{2,3}}{dh} = \frac{\sin \theta \cos \theta}{h^2 \sin^2 \theta + \cos^2 \theta}.$$

Substitute these equations to Equation (6.1) and we have

$$\frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left(-\frac{d\theta_{1,2}}{dh} \right) \left(\log \frac{2}{\sqrt{h^2 - 1}} - \ell_{0,1} \right) \\ = \frac{1}{2} \left(-\frac{d\theta_{1,2}}{dh} \right) \left(-\frac{1}{2} \log(h^2 - 1) + \log 2 - \ell_{0,1} \right).$$

Since

$$\lim_{h \downarrow 1} \left(-\frac{d\theta_{1,2}}{dh} \right) = \sin \theta \cos \theta, \quad \lim_{h \downarrow 1} \log(h^2 - 1) = -\infty,$$

we have $\lim_{h \downarrow 1} dV_{r,\theta}(h)/dh = +\infty$ in this case.

The uniqueness of the maximal value of $V_{r,\theta}$ with respect to h is obtained by the facts that $\log(h^2 - 1)$ is a strictly monotonic function and that $d\theta_{1,2}/dh \neq 0$ on $(1, +\infty)$.

Case (3): double frustums or Lambert cubes, i.e., $r > 1$. Since our strategy of proving that $V_{r,\theta}$ attains maximal on $(1, +\infty)$ is to prove that the limit of $dV_{r,\theta}(h)/dh$ is positive as h approaches to 1 from above, it is enough to consider the case that h is close enough to 1, meaning that $R_{r,\theta}(h)$ are double frustum, not Lambert cubes.

Under this assumption, use Equation (2.4) and we have

$$\begin{aligned}
\ell_{0,3} &= \operatorname{arccosh}(-\langle \mathbf{p}_0, \mathbf{p}_3 \rangle) \\
&= \operatorname{arccosh} \frac{1}{\sqrt{r^2-1} \sqrt{h^2-1}} \\
&= \log \left(\frac{1}{\sqrt{r^2-1} \sqrt{h^2-1}} + \sqrt{\left(\frac{1}{\sqrt{r^2-1} \sqrt{h^2-1}} \right)^2 - 1} \right) \\
&= \log \frac{\sqrt{(1-r^2)h^2 + r^2} + 1}{\sqrt{r^2-1} \sqrt{h^2-1}}.
\end{aligned}$$

Substitute this equation together with Equations (6.3) and (6.4) to Equation (6.1) and we have

$$(6.8) \quad \frac{dV_{r,\theta}(h)}{dh} = \frac{1}{2} \left(-\frac{d\theta_{1,2}}{dh} \right) \left(F(h) - \frac{1}{2} \log(r^2-1) \right),$$

where F is the function defined in Case (1).

By the same reason explained in Case (1), we have $\lim_{h \downarrow 1} dV_{r,\theta}(h)/dh = +\infty$ in this case as well.

We next prove that $V_{r,\theta}$ does not attain maximal when $R_{r,\theta}(h)$ is a Lambert cube, i.e., $h \in (r/\sqrt{r^2-1}, +\infty)$. What we actually prove is that $V_{r,\theta}$ is strictly decreasing, using Equation (6.1). Recall that $\ell_{0,3}$ is the hyperbolic distance between the polar geodesic plane of $\mathcal{P}(\mathbf{u}_1)$ and $\mathcal{P}(\mathbf{u}_2)$, and $\theta_{1,2}$ is the hyperbolic dihedral angle between the polar geodesic planes of P_0 and P_3 , while $\ell_{0,1}$ and $\theta_{2,3}$ are the same as in other cases. Using Equations (2.4) and (2.3), we have

$$\begin{aligned}
\ell_{0,3} &= \operatorname{arccosh}(-\langle \mathbf{u}_1, \mathbf{u}_2 \rangle) \\
&= \operatorname{arccosh} \frac{h \sin \theta}{\sqrt{(1-r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} \\
&= \log \frac{h \sin \theta + \sqrt{(r^2-1)h^2 - r^2} \cos \theta}{\sqrt{(1-r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}}, \\
\theta_{1,2} &= \arccos(-\langle \mathbf{p}_0, \mathbf{p}_3 \rangle) \\
&= \arccos \frac{1}{\sqrt{r^2-1} \sqrt{h^2-1}}, \\
\frac{d\theta_{1,2}}{dh} &= \frac{h}{(h^2-1) \sqrt{(r^2-1)h^2 - r^2}}.
\end{aligned}$$

The value $d\theta_{1,2}/dh$ is positive on $(r/\sqrt{r^2-1}, +\infty)$ by this expression, so is $d\theta_{2,3}/dh$ by Equation (6.4). The value $\ell_{0,1}$ is positive, for it is the hyperbolic length of an edge. By substituting these results to Equation (6.1), if we can prove that $\ell_{0,3} > 0$, then we have $dV_{r,\theta}/dh < 0$, namely $V_{r,\theta}$ is strictly decreasing, on $(r/\sqrt{r^2-1}, +\infty)$.

The inequality $\ell_{0,3} > 0$ is equivalent to

$$\frac{h \sin \theta + \sqrt{(r^2-1)h^2 - r^2} \cos \theta}{\sqrt{(1-r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} > 1.$$

Calculating

$$\left(\frac{h \sin \theta + \sqrt{(r^2 - 1) h^2 - r^2} \cos \theta}{\sqrt{(1 - r^2 \cos^2 \theta) h^2 + r^2 \cos^2 \theta}} \right)^2 - 1$$

and we have an inequality

$$\sqrt{(r^2 - 1) h^2 - r^2} h \sin \theta > - \{ (r^2 - 1) h^2 - r^2 \} \cos \theta,$$

which is equivalent to the previous one. This inequality holds on $(r/\sqrt{r^2 - 1}, +\infty)$, for the right-hand side is negative while the left hand side is positive. We have thus proved that $V_{r,\theta}$ does not attain maximal when $R_{r,\theta}(h)$ is a Lambert cube.

Since $V_{r,\theta}$ does not attain maximal when $R_{r,\theta}(h)$ is a Lambert cube, for the proof of the uniqueness of the maximal value of $V_{r,\theta}$, we can assume that $h \in (1, r/\sqrt{r^2 - 1}]$. Under this assumption together with the fact that $d\theta_{1,2}/dh \neq 0$ on $(1, r/\sqrt{r^2 - 1})$ by Equation (6.3), what we need to prove is that the number of elements in the set

$$\begin{aligned} \left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \mid \frac{dV_{r,\theta}(h)}{dh} = 0 \right\} \\ = \left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \mid F(h) - \frac{1}{2} \log(r^2 - 1) = 0 \right\} \end{aligned}$$

is at most one, where $dV_{r,\theta}(h)/dh$ is calculated in Equation (6.8) and the function F is given in Equation (6.5).

By Equation (6.7), we have

$$\frac{d}{dh} \left(F(h) - \frac{1}{2} \log(r^2 - 1) \right) = - \frac{h}{(h^2 - 1) \sqrt{(1 - r^2) h^2 + r^2}} G(h),$$

where we recall that $G(h) = C(1 - r^2)(h^2 - 1) + 1$. Unlike Case (1), the sign of the function G is not expected to be constant on $(1, r/\sqrt{r^2 - 1})$, for $1 - r^2 < 0$.

Since

$$\frac{h}{(h^2 - 1) \sqrt{(1 - r^2) h^2 + r^2}} \neq 0$$

on $(1, r/\sqrt{r^2 - 1})$, we have

$$\left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \mid F'(h) = 0 \right\} = \left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \mid G(h) = 0 \right\}.$$

The function G is quadratic with respect to h , the coefficient of h^2 is negative and $G(1) > 0$. These imply that the number of elements in the set of the right-hand side of the equation above is at most one, so is the set of the left-hand side of the equation.

Suppose that the number of elements in the set

$$\left\{ h \in (1, \frac{r}{\sqrt{r^2 - 1}}) \mid F(h) - \frac{1}{2} \log(r^2 - 1) = 0 \right\}$$

is more than 1. By the mean-value theorem together with the fact that the limit of $F(h) - (1/2) \log(r^2 - 1)$ is 0 as h approaches $r/\sqrt{r^2 - 1}$ from below, the set $\{ h \in (1, r/\sqrt{r^2 - 1}) \mid F'(h) = 0 \}$ must contain at least two elements, which contradicts the result obtained above.

We have thus proved Theorem 1. □

APPENDIX A. THE MAXIMAL AREA OF TWO-DIMENSIONAL HYPERBOLIC
COMPLETE ORTHOSCHEMES

By the definition of orthoscheme, a triangle $P_0P_1P_2$ in the two-dimensional hyperbolic space is orthoscheme if the edge P_0P_1 is perpendicular to the edge P_1P_2 , namely $P_0P_1P_2$ is a right-angled triangle with the right angle at P_1 . Without loss of generality, we suppose that $P_0P_1P_2$ lies in the projective disc model B^2 with the coordinates

$$P_0 = (r, 0), \quad P_1 = (0, 0), \quad P_2 = (0, h).$$

For a given $r > 0$, we consider a family $\{R_r(h)\}_{h>0}$ of complete orthoschemes, where $R_r(h)$ is a complete orthoscheme with vertices P_0 , P_1 and P_2 . What we discuss is the maximal area for this family.

Theorem 2. *The maximal area for $\{R_r(h)\}_{h>0}$ is obtained as follows:*

- (1) *For any $r < 1$, the area of $R_r(h)$ attains maximal just for $h = 1$. The maximal area is $\pi/2 - a(1)$, where $a(1)$ is the hyperbolic angle at P_0 of $R_r(1)$.*
- (2) *The area of $R_1(h)$ attains maximal for any $h \in [1, +\infty)$. The maximal area is $\pi/2$.*
- (3) *For any $r > 1$, the area of $R_r(h)$ attains maximal for any $h \in [1, r/\sqrt{r^2 - 1}]$. The maximal area is $\pi/2$.*

Proof. We start by recalling a formula to calculate the area A of a hyperbolic convex n -gon with hyperbolic angles $\alpha_1, \alpha_2, \dots, \alpha_n$;

$$A = (n - 2)\pi - (\alpha_1 + \alpha_2 + \dots + \alpha_n).$$

See Theorem 3.5.5 of [Ra] for the proof when $n = 3$.

Let $A_r(h)$ be the area of $R_r(h)$. For any $r > 0$, a complete orthoscheme $R_r(h)$ increases as a set when h approaches 1 from below, which implies that $A_r(h)$ also increases. So, to prove the theorem, it is enough to assume that $h \geq 1$. Using this formula, we obtain the area of $R_r(h)$ for each case.

- (1) Suppose $r < 1$. Let $a(h)$ be the hyperbolic angle at P_0 of $R_r(h)$.

When $h = 1$, $R_r(1)$ is a triangle with ideal vertex P_2 . Since the hyperbolic angle at P_2 is 0, the area is

$$\begin{aligned} A_r(1) &= \pi - \left(a(1) + \frac{\pi}{2} + 0\right) \\ &= \frac{\pi}{2} - a(1). \end{aligned}$$

When $h > 1$, $R_r(h)$ is a quadrilateral. The hyperbolic angles at the vertices constructed by truncation with respect to P_2 are right angles. The area is

$$\begin{aligned} A_r(h) &= 2\pi - \left(a(h) + \frac{\pi}{2} + \left(\frac{\pi}{2} + \frac{\pi}{2}\right)\right) \\ &= \frac{\pi}{2} - a(h). \end{aligned}$$

When h approaches $+\infty$, the corner at P_0 increases as a set, so is the angle $a(h)$. This implies that $A_r(h)$ is a strictly decrease function on $[1, +\infty)$.

As a result, $A_r(h)$ attains maximal if and only if $h = 1$ in this case.

- (2) Suppose $r = 1$. The hyperbolic angle at P_0 is 0 in this case. Use the argument in (1) with $a(h) = 0$ for any $h \geq 1$ and we have the desired conclusion.
- (3) Suppose $r > 1$.

When $h = 1$, $R_r(1)$ is a quadrilateral with angle 0 at P_2 and three right angles. The area is

$$\begin{aligned} A_r(1) &= 2\pi - \left(\left(\frac{\pi}{2} + \frac{\pi}{2} \right) + \frac{\pi}{2} + 0 \right) \\ &= \frac{\pi}{2}. \end{aligned}$$

When $h > 1$, there are two kinds of $R_r(h)$, which correspond to double frustums and Lambert cubes of three-dimensional complete orthoschemes.

- If $h < r/\sqrt{r^2 - 1}$, then $R_r(h)$ is a right-angled pentagon. The area is

$$\begin{aligned} A_r(h) &= 3\pi - \frac{\pi}{2} \times 5 \\ &= \frac{\pi}{2}. \end{aligned}$$

- If $h \geq r/\sqrt{r^2 - 1}$, then $R_r(h)$ is a quadrilateral, whose edges consists of P_0P_1 , P_1P_2 and polar lines of P_0 and P_2 . Let b be the hyperbolic angle between these polar lines. Then the area is

$$\begin{aligned} A_r(h) &= 2\pi - \left(\frac{\pi}{2} \times 3 + b \right) \\ &= \frac{\pi}{2} - b. \end{aligned}$$

The maximal area arises when $b = 0$, which occurs if and only if the polar planes of P_0 and P_2 are parallel, namely $h = r/\sqrt{r^2 - 1}$.

Summarizing these results, we have completed the proof. \square

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